

On the η -inverted sphere

Oliver Röndigs

September 19, 2016

Abstract

The first and second homotopy groups of the η -inverted sphere spectrum over a field of characteristic not two are zero. A cell presentation of higher Witt theory is given as well, at least over the complex numbers.

1 Introduction

Let $\eta: \mathbf{A}^2 \setminus \{0\} \rightarrow \mathbf{P}^1$ denote the first algebraic Hopf map given by sending a nonzero pair (x, y) to the line it generates in the plane. It induces an element of the same name $\eta \in \pi_{1,1}\mathbf{1}$ in the motivic stable homotopy groups of spheres. Hence η acts on the motivic stable homotopy groups of the sphere spectrum, in fact of any motivic spectrum. Inverting this action produces trivial groups in degrees one and two for the sphere spectrum.

Theorem 1.1 *Let F be a field of characteristic not two. Then*

$$\pi_{n+1,n}\mathbf{1}\left[\frac{1}{\eta}\right] = \pi_{n+2,n}\mathbf{1}\left[\frac{1}{\eta}\right] = 0$$

for every integer n .

This vanishing result enters the computation of the first line

$$\bigoplus_{n \in \mathbb{Z}} \pi_{n+1,n}\mathbf{1}$$

of motivic stable homotopy groups of spheres given in [31]. Fabien Morel identified the zero line of motivic stable homotopy groups of spheres, that is, the direct sum

$$\bigoplus_{n \in \mathbb{Z}} \pi_{n,n}\mathbf{1}$$

with the graded Milnor-Witt K -theory of the base field – see [24, Theorem 6.2.1], and the corresponding unstable statement [25, Theorem 4.9], documented in detail for perfect infinite fields in [26]. This identification implies that $\pi_{n,n}\mathbf{1}\left[\frac{1}{\eta}\right]$ is isomorphic to the Witt ring of the base field for every integer n . Impressive work of Guillou and Isaksen on the motivic Adams

spectral sequence at the prime two computes $\pi_{n+k,n}\mathbf{1}_2^\wedge[\frac{1}{\eta}]$ for all integers n and k over the real and complex numbers [11], [10]. It implies that $\pi_{n+3,n}\mathbf{1}_2^\wedge$ is nontrivial, at least over the real and the complex numbers. The next task is to determine $\pi_{n+3,n}\mathbf{1}[\frac{1}{\eta}]$ for an arbitrary field of characteristic not two, perhaps via the resolution of $\mathbf{1}[\frac{1}{\eta}]$ via connective Witt theory \mathbf{cKW} employed in Section 5. Only the first two maps

$$\mathbf{1}[\frac{1}{\eta}] \rightarrow \mathbf{cKW} \rightarrow \mathbf{cKW} \wedge \mathbf{cKW} \quad (1)$$

of this resolution enter the proof of Theorem 1.1. The zero line computation of Morel cited above allows to deduce the vanishing in Theorem 1.1 from the connectivity of the second map in (1), as Section 5 explains. This connectivity is obtained in Section 7, at least after inverting 2, by transferring results from topology on connective real K -theory to connective Witt theory. Proving that this transfer works proceeds by a real version of Joseph Ayoub's model for the Betti realization, to be explained in Section 3 and exploited in Section 4. The argument sketched so far would provide Theorem 1.1 only after inverting 2^1 , which annihilates Witt theory for fields which are not formally real – see Section 6. The integral computation requires knowledge on 2-complete computations, supplied by results from [31] with the help of the convergence result [17, Theorem 1]. The passage from 2-inverted to integral results is explained in Section 8.

Theorem 1.1 implies that a cell presentation of connective Witt theory starts by attaching 4-cells via the generators

$$\Sigma^{3,0}\mathbf{1}[\frac{1}{\eta}] \rightarrow \mathbf{1}[\frac{1}{\eta}]$$

of $\pi_{3,0}\mathbf{1}[\frac{1}{\eta}]$. For the complex numbers, where complete information is available by [11], a very small and complete cell presentation of connective Witt theory is obtained at the end of Section 4 (see also [14]). This cell presentation is not used in the proof of Theorem 1.1, but included for comparison with the impressive rational result [2]. Cell presentations and connectivity are discussed in Section 2. The following notation is used throughout.

F, S	field, finite dimensional separated Noetherian base scheme
\mathbf{Sm}_S	smooth schemes of finite type over S
$\mathbf{SH}(S)$	the motivic stable homotopy category of S
$\mathbf{E}, \mathbf{1} = S^{0,0}$	generic motivic spectrum, the motivic sphere spectrum
$S^{s,t}, \Sigma^{s,t}$	motivic (s, t) -sphere, (s, t) -suspension
$\pi_{s,t}\mathbf{E} = [S^{s,t}, \mathbf{E}]$	motivic stable homotopy groups of \mathbf{E}
$\pi_{s,t}\mathbf{E}$	sheaves of motivic stable homotopy groups of \mathbf{E}
\mathbf{KQ}, \mathbf{KW}	hermitian K -theory, Witt-theory

The suspension convention is such that $\mathbf{P}^1 \simeq S^{2,1}$ and $\mathbf{A}^1 \setminus \{0\} \simeq S^{1,1}$.

A previous version of this work was included in a preliminary version of [31], but separated for the sake of presentation. Theorem 1.1 enters only [31, Section 5], and its proof requires results from [31, Sections 2–4] – see Section 8 for details. I thank the RCN program *Topology in Norway*, and in particular the University of Oslo and Paul Arne Østvær, as well as the

¹See [6, Proposition 36] for a different proof of this vanishing statement after inverting 2.

DFG priority program *Homotopy theory and algebraic geometry* for support. I thank Joseph Ayoub for his input regarding a real version of his model of complex Betti realization. Tom Bachmann, Jeremiah Heller, and an anonymous referee receive my thanks for helpful comments.

2 Cellularity and connectivity

The following definitions regarding cells are modelled on [8] and [17]. *Attaching a cell* to a motivic spectrum E refers to the process of forming the pushout of

$$D^{s+1,t}\mathbf{1} \hookleftarrow \Sigma^{s,t}\mathbf{1} \xrightarrow{f} E \quad (2)$$

for some map f in the category of motivic spectra. The arrow pointing to the left in (2) denotes the canonical inclusion into $D^{s+1,t}\mathbf{1}$, the simplicial mapping cylinder of the map $\Sigma^{s,t}\mathbf{1} \rightarrow *$. The pushout D then consists of E , together with a cell of *dimension* $(s+1, t)$ and *weight* t . More generally, one may attach a collection of cells indexed by some set I by forming the pushout of a diagram

$$\bigvee_{i \in I} D^{s_i, t_i} \mathbf{1} \hookleftarrow \bigvee_{i \in I} \Sigma^{s_i, t_i} \mathbf{1} \xrightarrow{f} E$$

in the category of motivic spectra. A *cell presentation* of a map $f: D \rightarrow E$ of motivic spectra consists of a sequence of motivic spectra

$$D = D_{-1} \xrightarrow{d_0} D_0 \xrightarrow{\sim} D'_0 \xrightarrow{d_1} D_1 \xrightarrow{\sim} D'_1 \rightarrow \cdots \xrightarrow{d_n} D_n \xrightarrow{\sim} D'_n \xrightarrow{d_{n+1}} \cdots \quad (3)$$

with canonical map to the colimit $c: D \rightarrow D_\infty$, attaching maps

$$\bigvee_{i \in I_n} \Sigma^{s_i, t_i} \mathbf{1} \xrightarrow{\alpha_n} D'_{n-1}$$

for every natural number n , such that D_n is obtained by attaching cells to D'_{n-1} along α_n , and a weak equivalence $w: D_\infty \xrightarrow{\sim} E$ with $w \circ c = f$. Furthermore, all arrows in diagram (3) labelled with \sim denote acyclic cofibrations.

Remark 2.1 The most important instance is the absolute case, that is, a cell presentation of a map $* \rightarrow E$. Then one speaks of a cell presentation of E . The distinction between a motivic spectrum E and a cell presentation D_∞ of it might be neglected if no confusion can arise.

Example The suspension spectrum $\Sigma^\infty \mathbf{P}^\infty$ admits a cell presentation, having precisely one cell of dimension $(2n, n)$ for every natural number n .

Let $\mathbf{SH}(S)^{\text{cell}}$ denote the full subcategory of $\mathbf{SH}(S)$ of motivic spectra admitting a cell presentation. It can be identified with the homotopy category of *cellular* motivic spectra, that is, the smallest full localizing subcategory of $\mathbf{SH}(S)$ containing the spheres $S^{s,t}$, as [8, Remark 7.4] and the following statement show.

Lemma 2.2 *Let \mathbf{E} be a motivic spectrum. If it admits a cell presentation, it is cellular. Conversely, if it is cellular, it admits a cell presentation.*

Proof The first statement follows from the definitions. For the second statement, let \mathbf{E} be a cellular motivic spectrum which is fibrant. One constructs inductively a suitable sequence of motivic spectra. Start with

$$D_0 := \bigvee_{\alpha \in \pi_{s,t} \mathbf{E}, s, t} \Sigma^{s,t} \mathbf{1} \xrightarrow{\vee \alpha} \mathbf{E}$$

and factor the canonical map $D_0 \rightarrow \mathbf{E}$ as an acyclic cofibration $D_0 \xrightarrow{\sim} D'_0$ followed by a fibration $D'_0 \rightarrow \mathbf{E}$. The latter induces a surjection on $\pi_{s,t}$ for all s, t by construction. For every s, t , choose lifts of generators of the kernel of $\pi_{s,t}(D'_0 \rightarrow \mathbf{E})$ and use these to attach cells to D'_0 , leading to a map $D_1 \rightarrow \mathbf{E}$. As before, factor it as an acyclic cofibration, followed by a fibration $D'_1 \rightarrow \mathbf{E}$. This fibration still induces a surjection on $\pi_{s,t}$ for all s, t . Iterating this procedure leads to a map

$$D_\infty = \operatorname{colim}_n D_n \rightarrow \mathbf{E}$$

which induces a surjective and injective map on $\pi_{s,t}$ for all s, t . The statement on injectivity requires that $\Sigma^{s,t} \mathbf{1}$ is compact, whence any element in the kernel lifts to a finite stage and hence is killed in the next stage. Since both D_∞ and \mathbf{E} are cellular, the map $D_\infty \rightarrow \mathbf{E}$ is even a weak equivalence [8, Cor. 7.2].

The motivic stable homotopy category $\mathbf{SH}(S)$ is equipped with the *homotopy t -structure*

$$\mathbf{SH}(S)_{\geq n} \hookrightarrow \mathbf{SH}(S) \hookleftarrow \mathbf{SH}(S)_{\leq n}$$

where $\mathbf{SH}(S)_{\geq n} = \langle \Sigma^{s,t} \Sigma^\infty X_+ \mid X \in \mathbf{Sm}_S, s - t \geq n \rangle$ is the full subcategory generated under homotopy colimits and extensions by the shifted motivic suspension spectra of smooth S -schemes of connectivity at least n . See [16, Section 2.1] for details, and in particular [16, Theorem 2.3] for the identification with Morel's original definition via Nisnevich sheaves of motivic stable homotopy groups in case S is the spectrum of a field.

3 Real realization

Real Betti realization will be employed in order to give a topological interpretation of connective Witt theory over the real numbers. It is defined as the homotopy-colimit preserving functor $\mathbf{SH}(\mathbb{R}) \rightarrow \mathbf{SH}$ from the motivic stable homotopy category of the real numbers to the classical motivic stable homotopy category which is determined by sending the suspension spectrum of a smooth variety X over \mathbb{R} to the suspension spectrum of the topological space of real points $X(\mathbb{R})$, equipped with the real analytic topology. Another viewpoint on the real Betti realization is given by equivariant stable homotopy theory with respect to the absolute Galois group C_2 of \mathbb{R} : The real Betti realization is the composition of geometric fixed points and the complex Betti realization; see [13, Section 4.4] for details. Real Betti realization is given already unstably on the level of presheaves on $\mathbf{Sm}_{\mathbb{R}}$ with values in simplicial sets, and

as such it is a simplicial functor. There is an additive variant for presheaves with values in simplicial abelian groups, or, equivalently, complexes of abelian groups.

Theorem 3.1 *Real Betti realization $\mathbb{R}^*: \mathbf{SH}(\mathbb{R}) \rightarrow \mathbf{SH}$ is strict symmetric monoidal and has a right adjoint \mathbb{R}_* . Moreover, the canonical map*

$$D \wedge \mathbb{R}_*(E) \rightarrow \mathbb{R}_*(\mathbb{R}^*(D) \wedge E) \quad (4)$$

is an equivalence for all motivic spectra $D \in \mathbf{SH}(\mathbb{R})$.

Proof The first statement follows from [27, Section 3.3], and also from [13, Proposition 4.8]. The second statement follows from [9, Proposition 3.2] for strongly dualizable motivic spectra. Since $\mathbf{SH}(\mathbb{R})$ is generated by strongly dualizable motivic spectra, the result follows.

Identifying the real Betti realization of motivic spectra which are not suspension spectra of smooth varieties is not immediate. However, Joseph Ayoub's beautiful model for the complex Betti realization given in [4, Théorème 2.67] translates to a model for the real Betti realization. In order to define this model, let \mathcal{I}^n denote the pro-real analytic manifold obtained by open neighborhoods of the compact unit cube $[0, 1]^n \subset \mathbb{R}^n$. Letting n vary defines, in the standard way, a cocubical pro-real analytic manifold \mathcal{I}^\bullet as defined in [4, Def. A.1]. One observes that \mathcal{I}^\bullet is in fact a pseudo-monoidal Σ -enriched cocubical pro-real analytic manifold [4, Def. A.29]. Let \mathcal{R}_n denote the $\mathbb{R}[t_1, \dots, t_n]$ -algebra of real analytic functions defined on an open neighborhood of $[0, 1]^n$.

Proposition 3.2 (Ayoub) *The $\mathbb{R}[t_1, \dots, t_n]$ -algebra \mathcal{R}_n is Noetherian and regular.*

Proof The $\mathbb{C}[t_1, \dots, t_n]$ -algebra $\mathcal{R}_n \otimes_{\mathbb{R}} \mathbb{C}$ is the algebra of complex analytic functions defined on an open neighborhood of $[0, 1]^n$, which is Noetherian and regular. The required statement follows by Galois descent.

Popescu's theorem [34, Theorem 10.1] then implies that \mathcal{R}_n is a filtered colimit of smooth $\mathbb{R}[t_1, \dots, t_n]$ -algebras, and thus can be considered as an affine pro-smooth \mathbb{R} -variety $\mathrm{Spec}(\mathcal{R}_n)$. Hence any presheaf K on $\mathbf{Sm}_{\mathbb{R}}$ admits a value $K(\mathrm{Spec}(\mathcal{R}_n))$. Letting n vary defines a cubical object $K(\mathrm{Spec}(\mathcal{R}_\bullet))$.

Theorem 3.3 (Ayoub) *Let $C: \mathbf{Sm}_{\mathbb{R}} \rightarrow \mathbf{Cx}$ be a presheaf of complexes of abelian groups. The real Betti realization of C is quasi-isomorphic to the total complex $C(\mathrm{Spec}(\mathcal{R}_\bullet))$.*

In fact, as in the complex case given in [4, Théorème 2.61] it is possible to replace $\mathrm{Spec}(\mathcal{R}_n)$ in Theorem 3.3 by $\mathrm{Spec}(\mathcal{R}_n^{et})$ where \mathcal{R}_n^{et} is, roughly speaking, the largest sub-algebra of \mathcal{R}_n which is pro-étale over $\mathbb{R}[t_1, \dots, t_n]$.

4 Witt theory

Let KQ denote the motivic spectrum for hermitian K -theory of quadratic forms [15]. It is defined over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$, and hence over any scheme in which 2 is invertible. Note the

identification

$$\mathbf{KQ}_{s,t} = \pi_{s,t} \mathbf{KQ} = GW_{s-2t}^{[-t]}(F)$$

with Schlichting's higher Grothendieck-Witt groups [33, Definition 9.1]. The periodicity element is denoted $\alpha: S^{8,4} \rightarrow \mathbf{KQ}$. It is denoted β in [1].

Theorem 4.1 *Suppose S is a scheme over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$. Then \mathbf{KQ} is a cellular commutative motivic ring spectrum in $\mathbf{SH}(S)$ which is preserved under base change.*

Proof A ring structure is provided in [28]. Cellularity follows basically from the model given in [28]; details may be found in [30]. The statement regarding base change refers to the fact that, given a morphism $f: S \rightarrow S'$ of schemes over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$, there is a canonical identification $f^*(\mathbf{KQ}) \rightarrow \mathbf{KQ}$ of motivic spectra over S . It is induced by the corresponding canonical identification of Grassmannians which serve to model \mathbf{KQ} by [28].

Let \mathbf{KW} be the motivic ring spectrum representing higher Balmer-Witt groups in the motivic stable homotopy category. It can be described as $\mathbf{KW} = \mathbf{KQ}[\frac{1}{\eta}]$, where $\eta: S^{1,1} \rightarrow S^{0,0}$ is the first algebraic Hopf map.

Corollary 4.2 *Suppose S is a scheme over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$. Then \mathbf{KW} is a cellular motivic ring spectrum in $\mathbf{SH}(S)$ which is preserved under base change. It is a commutative $\mathbf{1}[\frac{1}{\eta}]$ -algebra.*

Over fields, one may describe the coefficients of \mathbf{KW} as follows:

$$\mathbf{KW}_{*,*} \cong \mathbf{KW}_{0,0}[\eta, \eta^{-1}, \alpha, \alpha^{-1}] \cong W[\eta, \eta^{-1}, \alpha, \alpha^{-1}] \quad (5)$$

Here W is the Witt ring of the base field and α is the following composition:

$$S^{8,4} \xrightarrow{\alpha} \mathbf{KQ} \xrightarrow{\text{can.}} \mathbf{KW}$$

The remainder of the section will be devoted to specific descriptions of the motivic spectrum \mathbf{KW} , after suitable modifications, or over specific fields. See [32, Theorem 2.6.4] for the following statement.

Theorem 4.3 (Pfister) *Let S be the spectrum of a field. The additive group underlying the graded ring $\mathbf{KW}_{*,*}[\frac{1}{2}]$ is torsion-free.*

In order to state the next theorem, let \mathbf{KO} denote the topological spectrum representing real topological K -theory. Its 0-connective cover is denoted $\mathbf{k}\mathbf{o}$. The coefficients of the topological spectrum $\mathbf{KO}[\frac{1}{2}]$ are particularly simple: A Laurent polynomial ring on a single generator α_{top} of degree 4.

Theorem 4.4 (Brumfiel) *There is an equivalence $\mathbb{R}^* \mathbf{KW}[\frac{1}{2}] \simeq \mathbf{KO}[\frac{1}{2}]$ of topological spectra. The unit map $\mathbf{KW}[\frac{1}{2}] \rightarrow \mathbb{R}_* \mathbf{KW}[\frac{1}{2}]$ is an equivalence in $\mathbf{SH}(\mathbb{R})$.*

Proof This follows from [19, Theorem 6.2] (but see also [2, Remark 3]) which provides a natural equivalence of classical spectra

$$\mathrm{KW}_0[\tfrac{1}{2}](X) \rightarrow \mathrm{KO}[\tfrac{1}{2}](\mathbb{R}^* X)$$

for every finite type \mathbb{R} -scheme X . The Yoneda lemma then implies that the map induced by the functor \mathbb{R}^* on internal simplicial sets of morphisms is an equivalence for all smooth \mathbb{R} -schemes, whose $\Sigma^{s,t}$ -suspensions generate $\mathbf{SH}(\mathbb{R})$. Adjointness then implies that the unit map $\mathrm{KW}[\tfrac{1}{2}] \rightarrow \mathbb{R}_* \mathbb{R}^* \mathrm{KW}[\tfrac{1}{2}]$ is an equivalence in $\mathbf{SH}(\mathbb{R})$.

Due to the α -periodicity of its target, the unit $\mathbf{1} \rightarrow \mathrm{KW}$ is not k -connective for any $k \in \mathbb{Z}$. Let $\psi: \mathbf{cKW} := \mathrm{KW}_{\geq 0} \rightarrow \mathrm{KW}$ denote the 0-connective cover with respect to the homotopy t -structure. Since $\mathbf{1}$ is 0-connective, the $\mathbf{1}[\tfrac{1}{\eta}]$ -algebra homomorphism $u: \mathbf{1}[\tfrac{1}{\eta}] \rightarrow \mathrm{KW}$ factors uniquely as a $\mathbf{1}[\tfrac{1}{\eta}]$ -algebra homomorphism $cu: \mathbf{1}[\tfrac{1}{\eta}] \rightarrow \mathbf{cKW}$, even in a strict sense if one uses [12] whose general setup applies here. The next aim is to provide an analog of Theorem 4.4 for \mathbf{cKW} instead of KW .

Theorem 4.5 *Suppose S is a scheme over $\mathrm{Spec}(\mathbb{Z}[\tfrac{1}{2}])$. Then \mathbf{cKW} is a commutative $\mathbf{1}[\tfrac{1}{\eta}]$ -algebra which is preserved under base change.*

Proof The statement regarding the multiplicative structure follows from [12]. The base change argument for connective covers is given in [16, Lemma 2.2].

Lemma 4.6 *The canonical map $\mathbf{cKW} \rightarrow \mathrm{KW}$ coincides with the canonical map $\mathbf{cKW} \rightarrow \mathbf{cKW}[\alpha^{-1}]$ up to canonical equivalence.*

Proof By construction, $\alpha: S^{8,4} \rightarrow \mathrm{KW}$ is an invertible element. Moreover, it lifts to a map $S^{8,4} \rightarrow \mathbf{cKW}$ deserving the same notation. Hence there is a canonical map $\mathbf{cKW}[\alpha^{-1}] \rightarrow \mathrm{KW}$ which induces an isomorphism on sheaves of homotopy groups, whence the statement.

Lemma 4.6 implies that the filtration on KW given by the homotopy t -structure coincides with multiplications by powers of α on suspensions of \mathbf{cKW} . Thus the cone of multiplication by α on \mathbf{cKW} , henceforth denoted \mathbf{cKW}/α , coincides with the 0-truncation of KW , or equivalently, with the motivic Eilenberg-MacLane spectrum for the sheaf of unramified Witt groups. In particular, over a field F of characteristic not two, it is a motivic spectrum whose homotopy groups are concentrated in the zero line:

$$\pi_{s,t} \mathbf{cKW}/\alpha \cong \begin{cases} W(F) & s = t \\ 0 & s \neq t \end{cases}$$

Moreover, the canonical map $\mathbf{cKW} \rightarrow \mathbf{cKW}/\alpha$ is even a map of $\mathbf{1}[\tfrac{1}{\eta}]$ -algebras. The homotopy t -structure filtration of KW thus consists of \mathbf{cKW} -modules, and its associated graded consists of (de)suspensions of \mathbf{cKW}/α .

Lemma 4.7 *The canonical map $\mathbb{R}^*(\mathbf{cKW}/\alpha) \rightarrow H\mathbb{Z}[\tfrac{1}{2}]$ is an equivalence of topological spectra.*

Proof The algebraic Hopf map η induces the degree 2 map on the topological sphere spectrum after taking real points. Morel's Theorem, see 5.2, implies the map $\mathbf{1}[\frac{1}{\eta}] \rightarrow \mathbf{cKW}$ is 1-connective and hence induces a 1-connective map

$$\mathbb{S}[\frac{1}{2}] = \mathbb{R}^*(\mathbf{1}[\frac{1}{\eta}]) \rightarrow \mathbb{R}^*\mathbf{cKW}$$

of topological spectra. This identifies $\pi_0\mathbb{R}^*\mathbf{cKW}$ as $\mathbb{Z}[\frac{1}{2}]$ (and $\pi_1\mathbb{R}^*\mathbf{cKW}$ as the trivial group). Since the canonical map $\mathbf{cKW} \rightarrow \mathbf{cKW}/\alpha$ is 4-connective, also $\pi_0\mathbb{R}^*\mathbf{cKW}/\alpha \cong \mathbb{Z}[\frac{1}{2}]$ (and $\pi_1\mathbb{R}^*\mathbf{cKW}/\alpha = 0$). The canonical map mentioned in the statement of the lemma is the map to the zeroth Postnikov section. It remains to show that $\pi_n\mathbb{R}^*\mathbf{cKW}/\alpha = 0$ for all $n \geq 2$. For this purpose, recall that \mathbf{cKW}/α is the Eilenberg-MacLane spectrum associated to the sheaf of unramified Witt groups on $\mathbf{Sm}_{\mathbb{R}}$. Hence its real Betti realization is determined by the real Betti realization of the sheaf W of Witt groups, considered as a complex concentrated in degree zero, and the spectrum structure maps. Theorem 3.3 allows to identify the real Betti realization of W as the complex $W(\mathrm{Spec}(\mathcal{R}_{\bullet}))$, which turns out to be the complex

$$\mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\mathrm{id}} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\mathrm{id}} \dots$$

by Lemma 4.8 below. The result follows.

The statement of Lemma 4.7 for $\mathbf{cKW}[\frac{1}{2}]/\alpha$ is mentioned below the proof of [5, Proposition 29]. Note that the proof of Lemma 4.7 supplies an integral identification of the real points of the S^1 -Eilenberg-MacLane spectrum for the sheaf of Witt groups.

Lemma 4.8 *The inclusions*

$$\mathbb{R} \hookrightarrow \mathbb{R}[t_1, \dots, t_n] \hookrightarrow \mathcal{R}_n \hookrightarrow \mathbb{R}[[t_1, \dots, t_n]]$$

induce isomorphisms on Witt groups for every n .

Proof By construction, $\mathcal{R}_0 = \mathbb{R}$. Shrinking the cube defines a filtered system of intermediate algebras between \mathcal{R}_n and the formal power series ring $\mathbb{R}[[t_1, \dots, t_n]]$ whose colimit is the ring of germs of real analytic functions at zero. The intermediate algebras are all isomorphic to \mathcal{R}_n . Moreover, the restriction homomorphism between any two such defines an isomorphism on Witt groups, as one concludes by a homotopy interpolating between the two cube diameters. Since the Witt group commutes with filtered colimits, the result follows.

Corollary 4.9 *There is an equivalence $\mathbb{R}^*\mathbf{cKW}[\frac{1}{2}] \simeq \mathbf{ko}[\frac{1}{2}]$ of topological spectra. The unit $\mathbf{cKW}[\frac{1}{2}] \rightarrow \mathbb{R}_*\mathbb{R}^*\mathbf{cKW}[\frac{1}{2}] \simeq \mathbb{R}_*(\mathbf{ko}[\frac{1}{2}])$ is an equivalence in $\mathbf{SH}(\mathbb{R})$.*

Proof This follows from Theorem 4.4, Lemma 4.7, and the compatibility of $(\mathbb{R}^*, \mathbb{R}_*)$ with the homotopy t -structure.

Over the complex numbers, a very explicit small cell presentation of \mathbf{cKW} can be given, thanks to the following fantastic theorem [11], [3]. This cell presentation will not be used in the remaining sections.

Theorem 4.10 (Andrews-Miller) *Over \mathbb{C} , the graded ring $\pi_{*,*}\mathbf{1}[\frac{1}{\eta}]$ is isomorphic to the ring $\mathbb{F}_2[\eta, \eta^{-1}, \sigma, \mu_9]/\eta\sigma^2$ where $|\eta| = (1, 1)$, $|\sigma| = (7, 4)$, and $|\mu_9| = (9, 5)$.*

A priori, the Andrews-Miller computation produces the homotopy groups of $\mathbf{1}[\frac{1}{\eta}]^\wedge$, but since $\pi_{0,0}\mathbf{1}[\frac{1}{\eta}] \cong \mathbb{Z}/2$ over \mathbb{C} by Morel's Theorem 5.1, $\mathbf{1}[\frac{1}{\eta}]$ is already 2-complete over \mathbb{C} . Theorem 4.10 implies the following statement on the slices of $\mathbf{1}[\frac{1}{\eta}]$, as explained in [31, Theorem 2.34].

Theorem 4.11 *Suppose S is a scheme over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$. There is a splitting of the zero slice of the η -inverted sphere spectrum*

$$s_0(\mathbf{1}[\frac{1}{\eta}]) \cong \bigvee_{1 \neq n \geq 0} \Sigma^{n,0} \mathbf{M}\mathbb{Z}/2.$$

such that the unit $\mathbf{1}[\frac{1}{\eta}] \rightarrow \mathbf{KW}$ induces the inclusion on every even summand.

The remainder of this section takes place in the category of $\mathbf{1}[\frac{1}{\eta}]$ -modules over the field of complex numbers. Abbreviate $\bigoplus_n \pi_{n+k,n}$ as π_k .

Lemma 4.12 *The unit $\mathbf{1}[\frac{1}{\eta}] \rightarrow \mathbf{cKW}$ induces an isomorphism $\pi_{4k}\mathbf{1}[\frac{1}{\eta}] \rightarrow \pi_{4k}\mathbf{cKW}$ for every integer k .*

Proof The unique nontrivial element $\alpha^k \eta^{-4k} \in \pi_{4k,0}\mathbf{KW}$ comes from the unique nontrivial element in the $4k$ th column of the $E^2 = E^\infty$ page of the zeroth slice spectral sequence of \mathbf{KW} [29]. The description of the unit map $\mathbf{1}[\frac{1}{\eta}] \rightarrow \mathbf{KW}$ on slices, Theorem 4.11, implies that this element is the image of the unique nontrivial element in the $4k$ th column of the $E^2 = E^\infty$ page of the zeroth slice spectral sequence of $\mathbf{1}[\frac{1}{\eta}]$. Another proof, which does not rely on Voevodsky's slice filtration, is given in [14, Theorem 3.2].

Set $D_1 = \mathbf{1}[\frac{1}{\eta}]$. Then the unit $\mathbf{1}[\frac{1}{\eta}] = D_1 \rightarrow \mathbf{cKW}$ is a 3-connective map.² Set D_2 to be the homotopy cofiber of

$$\sigma\eta^{-4}: \Sigma^{3,0}\mathbf{1}[\frac{1}{\eta}] \rightarrow \mathbf{1}[\frac{1}{\eta}] = D_1$$

Then the canonical map $\mathbf{1}[\frac{1}{\eta}] = D_1 \rightarrow D_2$ induces an isomorphism on π_m for $m \geq 4$, as a consequence of the long exact sequence of homotopy groups and Theorem 4.10. The unit $\mathbf{1}[\frac{1}{\eta}] \rightarrow \mathbf{cKW}$ then factors over D_2 as a 7-connective map $D_2 \rightarrow \mathbf{cKW}$ by Lemma 4.12.

This can be continued inductively. Suppose that there exists a sequence of cellular motivic spectra factoring the unit of \mathbf{cKW} as

$$\mathbf{1}[\frac{1}{\eta}] = D_1 \rightarrow D_2 \rightarrow \cdots \rightarrow D_n \rightarrow \mathbf{cKW}$$

where the last map is $4n - 1$ -connective, and where for every $k \leq n$

$$\mathbf{1}[\frac{1}{\eta}] \rightarrow D_k$$

²True over every field of characteristic not two, as Theorem 8.3 shows.

induces an isomorphism on π_m for $m \geq 4(k-1)$. Consider the (over \mathbb{C} unique nontrivial) element $\Sigma^{4n-1} \mathbf{1}[\frac{1}{\eta}] \rightarrow D_n$ in $\pi_{4n-1} D_n \cong \pi_{4n-1} \mathbf{1}[\frac{1}{\eta}]$ inducing an isomorphism on π_{4n-1} . Then its homotopy cofiber D_{n+1} satisfies the following two properties:

1. The map $\mathbf{1}[\frac{1}{\eta}] \rightarrow D_{n+1}$ induces an isomorphism on π_m for $m \geq 4n$.
2. The map $D_n \rightarrow \mathbf{cKW}$ factors over $D_n \rightarrow D_{n+1}$ as a $4n+3$ -connective map $D_{n+1} \rightarrow \mathbf{cKW}$.

The first property follows from the long exact sequence of homotopy groups, the condition on D_n , and Theorem 4.10. The second property then follows from the first property, equation (5), the condition on D_n , and Lemma 4.12. Taking the colimit with respect to $n \rightarrow \infty$ produces the desired cell presentation of \mathbf{cKW} by Lemma 4.12. A cell presentation for \mathbf{KW} then follows from Lemma 4.6. Rationally this cell presentation splits by [2], even over any field of characteristic not two.

5 An Adams resolution with connective Witt theory

The section title refers to the cosimplicial diagram

$$[n] \mapsto \mathbf{cKW}^{\wedge n+1}$$

determined by the $\mathbf{1}[\frac{1}{\eta}]$ -algebra \mathbf{cKW} . The starting point of this resolution of $\mathbf{1}[\frac{1}{\eta}]$ is the following.

Theorem 5.1 (Morel) *If F is a field, $\pi_{n,n} \mathbf{1}[\frac{1}{\eta}]$ is isomorphic to the Witt ring of F .*

It translates to the following statement.

Corollary 5.2 *Let F be a field of characteristic not two. The unit $cu: \mathbf{1}[\frac{1}{\eta}] \rightarrow \mathbf{cKW}$ is 1-connective.*

In other words, $\pi_{n,n} cu$ is an isomorphism and $\pi_{n+1,n} cu$ is surjective for every integer n . Let

$$\mathbb{C} \rightarrow \mathbf{1}[\frac{1}{\eta}] \xrightarrow{cu} \mathbf{cKW} \tag{6}$$

be the fiber of cu . Corollary 5.2 implies that it is 1-connective.

Lemma 5.3 *The canonical map*

$$\pi_{p,q} \mathbb{C} \rightarrow \pi_{p,q} \mathbf{1}[\frac{1}{\eta}]$$

is an isomorphism if $p - q \equiv 1, 2(4)$ and surjective if $p - q \equiv 3(4)$.

Proof The cofiber sequence (6) induces a long exact sequence of sheaves of homotopy groups. The result then follows from vanishing $\pi_{p,q} \mathbf{cKW} = 0$ for $p - q$ not divisible by 4, which in turn follows from the fact that higher Witt groups of a field are concentrated in degrees congruent to 0 modulo 4.

Smashing the cofiber sequence (6) with \mathbf{cKW} produces the following cofiber sequence:

$$\mathbf{C} \wedge \mathbf{cKW} \rightarrow \mathbf{1}_{[\frac{1}{\eta}]} \wedge \mathbf{cKW} = \mathbf{cKW} \xrightarrow{cu \wedge \mathbf{cKW}} \mathbf{cKW} \wedge \mathbf{cKW} \quad (7)$$

Lemma 5.4 *The connecting map*

$$\pi_{p+1,q} \mathbf{cKW} \wedge \mathbf{cKW} \rightarrow \pi_{p,q} \mathbf{C} \wedge \mathbf{cKW}$$

is an isomorphism for $p - q \equiv 1, 2(4)$, and surjective for $p - q \equiv 0, 3(4)$.

Proof The cofiber sequence (7) induces a long exact sequence of homotopy groups. As in the proof of Lemma 5.3, the vanishing $\pi_{p,q} \mathbf{cKW} = 0$ for $p - q$ not divisible by 4 implies the statement for $p - q \equiv 1, 2, 3(4)$. Since $cu \wedge \mathbf{cKW}$ has the multiplication as a retraction, surjectivity also holds for $p - q \equiv 0(4)$.

An explicit consequence of Lemma 5.4 is that $\pi_{n+1,n} \mathbf{cKW} \wedge \mathbf{cKW} = 0$ for all integers n . In fact, \mathbf{C} is 1-connective by Corollary 5.2, whence $\pi_{n,n} \mathbf{C} \wedge \mathbf{cKW} = 0$.

Proposition 5.5 *For every integer n , there is an isomorphism*

$$\pi_{n+2,n} \mathbf{cKW} \wedge \mathbf{cKW} \cong \pi_{n+1,n} \mathbf{1}_{[\frac{1}{\eta}]}$$

of sheaves of homotopy groups.

Proof Smashing the cofiber sequence (6) with \mathbf{C} produces the following cofiber sequence:

$$\mathbf{C} \wedge \mathbf{C} \rightarrow \mathbf{1}_{[\frac{1}{\eta}]} \wedge \mathbf{C} = \mathbf{C} \xrightarrow{cu \wedge \mathbf{C}} \mathbf{cKW} \wedge \mathbf{C}$$

Since \mathbf{C} is 1-connective by Morel's Theorem 5.2, $\mathbf{C} \wedge \mathbf{C}$ is 2-connective, which implies that $\pi_{n+1,n} \mathbf{C} \rightarrow \pi_{n+1,n} \mathbf{cKW} \wedge \mathbf{C}$ is an isomorphism. Lemma 5.4 gives that the connecting map $\pi_{n+2,n} \mathbf{cKW} \wedge \mathbf{cKW} \rightarrow \pi_{n+1,n} \mathbf{cKW} \wedge \mathbf{C}$ is an isomorphism. The map $\pi_{n+1,n} \mathbf{C} \rightarrow \pi_{n+1,n} \mathbf{1}$ is an isomorphism by Lemma 5.3. The appropriate composition provides the desired isomorphism.

Proposition 5.6 *If $\pi_{n+1,n} \mathbf{1}_{[\frac{1}{\eta}]} = 0$, then there is an isomorphism*

$$\pi_{n+3,n} \mathbf{cKW} \wedge \mathbf{cKW} \cong \pi_{n+2,n} \mathbf{1}_{[\frac{1}{\eta}]}$$

of sheaves of homotopy groups for every integer n .

Proof Assume that $\pi_{n+1,n} \mathbf{1}_{[\frac{1}{\eta}]} = 0$, or, equivalently, that \mathbf{C} is 2-connective. Then $\mathbf{C} \wedge \mathbf{C}$ is 4-connective, which implies that $\pi_{n+k,n} \mathbf{C} \rightarrow \pi_{n+k,n} \mathbf{cKW} \wedge \mathbf{C}$ is an isomorphism for all integers n and all integers $k \leq 3$, and in particular for $k = 2$. Lemma 5.4 implies that the connecting map $\pi_{n+3,n} \mathbf{cKW} \wedge \mathbf{cKW} \rightarrow \pi_{n+2,n} \mathbf{cKW} \wedge \mathbf{C}$ is an isomorphism. The canonical map $\pi_{n+2,n} \mathbf{C} \rightarrow \pi_{n+2,n} \mathbf{1}$ is an isomorphism by Lemma 5.3. The appropriate composition provides the desired isomorphism.

Propositions 5.5 and 5.6 are direct manifestations of the applicability of cooperations in connective Witt theory to computations of motivic stable homotopy groups of the η -inverted sphere. The following structural result has consequences for $\mathbf{cKW} \wedge \mathbf{cKW}$. In order to state it, recall the real étale topology on \mathbf{Sm}_S which has as stalks henselian local rings with real closed residue fields. In particular, it is finer than the Nisnevich topology. The identity functor on motivic (symmetric) spectra over S thus can be regarded as a left Quillen functor from the Nisnevich to the real étale homotopy theory, and similarly for the respective \mathbf{A}^1 -localizations. The real étale topology is relevant because of [19, Theorem 6.6], which implies that a Nisnevich fibrant model for $\mathbf{cKW}[\frac{1}{2}]$ is already real étale fibrant. Besides being crucial input for a proof of Theorem 4.4 above, this gives the base case for the following statement (which could be formulated in greater generality).

Proposition 5.7 *Let F be a field of characteristic zero. A Nisnevich fibrant $\mathbf{cKW}[\frac{1}{2}]$ -module over F is already real étale fibrant.*

Proof The standard model structure on modules over a ring object has underlying weak equivalences and fibrations. The corresponding model structure in the case at hand is cofibrantly generated. The cofibers of these generating cofibrations are free modules $\mathbf{cKW}[\frac{1}{2}] \wedge \mathbf{G}$, where \mathbf{G} is a shifted motivic suspension spectrum of a smooth F -scheme. A standard argument shows that it suffices to prove that a Nisnevich fibrant replacement of $\mathbf{cKW}[\frac{1}{2}] \wedge \mathbf{G}$ is already real étale fibrant. Since the motivic symmetric spectra \mathbf{G} are strongly dualizable, a Nisnevich fibrant replacement is given by the internal motivic spectrum of maps from (a cofibrant model of) the dual of \mathbf{G} to a Nisnevich fibrant model of $\mathbf{cKW}[\frac{1}{2}]$. However, since the latter is already real étale fibrant, so is the internal motivic spectrum of maps.

Proposition 5.7 assumes that 2 is invertible and that the base field has characteristic zero. Fields of positive characteristic do not have interesting Witt theory after inverting 2, as the following short section discusses for the sake of completeness.

6 Fields of odd characteristic

Theorem 6.1 *Let F be a field of odd characteristic. Then $\mathbf{1}[\frac{1}{\eta}, \frac{1}{2}]$ is contractible.*

Proof One argument (there are simpler ones) uses the beginning of the \mathbf{cKW} -resolution of $\mathbf{1}[\frac{1}{\eta}]$ sketched in Section 5. The motivic spectrum $\mathbf{cKW}[\frac{1}{2}]$ has trivial homotopy sheaves by [32, Theorem 2.6.4]. Hence it is contractible, and so is the motivic spectrum $\mathbf{C} \wedge \mathbf{cKW}[\frac{1}{2}]$, where \mathbf{C} is the homotopy fiber of $\mathbf{1}[\frac{1}{\eta}] \rightarrow \mathbf{cKW}$ described in Section 5. It follows that the canonical map $\mathbf{C} \wedge \mathbf{C}[\frac{1}{2}] \rightarrow \mathbf{C}[\frac{1}{2}]$ is a weak equivalence. However, since \mathbf{C} is 1-connective by Morel's Theorem 5.2, $\mathbf{C} \wedge \mathbf{C}$ is 2-connective. Hence $\mathbf{C}[\frac{1}{2}]$ is 2-connective, so $\mathbf{C} \wedge \mathbf{C}[\frac{1}{2}]$ is 4-connective, and so on. Thus $\mathbf{C}[\frac{1}{2}]$ is contractible, which implies the same for $\mathbf{1}[\frac{1}{\eta}, \frac{1}{2}]$.

Theorem 6.1 holds for any field whose Witt group is (necessarily 2-primary) torsion. This class of fields coincides with the class of non formally real fields [32, Theorem 2.7.1].

7 Witt cooperations

All statements on Witt cooperations will be deduced from the rational case [2] and the following topological result.

Theorem 7.1 (Mahowald, Kane, Lellmann) *For every prime p , there exists a sequence $B(j)$ of connective topological spectra starting with S^0 , a sequence s_j of natural numbers with $s_j \geq 4j$, and a p -local equivalence*

$$\gamma: \bigvee_{j \geq 0} \Sigma^{s_j} B(j) \wedge \mathbf{ko} \rightarrow \mathbf{ko} \wedge \mathbf{ko} \quad (8)$$

of topological spectra.

Proof This follows from [22] for the prime 2 (which is irrelevant for the following arguments), [18], and [21] for odd primes. See also [23] and [7].

Theorem 7.1 can be reinterpreted as follows. Fix an odd prime. Consider the motivic spectrum $\Sigma^{s_j, 0} B(j) \wedge \mathbf{cKW}[\frac{1}{2}]$ over the real numbers. Here the smash product with the topological spectrum $B(j)$ can be interpreted in two equivalent ways. One way is to view the category of motivic (symmetric) spectra as enriched over the category of usual (symmetric) spectra, and to use an appropriate simplicial version of $B(j)$. Another way, to be pursued here, is to consider $B(j)$ as a constant presheaf of usual S^1 -spectra, having a motivic suspension spectrum with the same notation. The real Betti realization of $\Sigma^{s_j, 0} B(j) \wedge \mathbf{cKW}[\frac{1}{2}]$ is

$$\mathbb{R}^*(\Sigma^{s_j, 0} B(j) \wedge \mathbf{cKW}[\frac{1}{2}]) \simeq \mathbb{R}^* \Sigma^{s_j, 0} B(j) \wedge \mathbb{R}^* \mathbf{cKW}[\frac{1}{2}] \simeq \Sigma^{s_j} B(j) \wedge \mathbf{ko}[\frac{1}{2}]$$

by Theorem 3.1 and Corollary 4.9. Composition with γ from (8) yields a map

$$\mathbb{R}^*(\Sigma^{s_j, 0} B(j) \wedge \mathbf{cKW}[\frac{1}{2}]) \rightarrow \mathbf{ko} \wedge \mathbf{ko}[\frac{1}{2}] \xrightarrow{\sim} \mathbb{R}^*(\mathbf{cKW} \wedge \mathbf{cKW}[\frac{1}{2}]) \quad (9)$$

whose adjoint is a map $\Sigma^{s_j, 0} B(j) \wedge \mathbf{cKW}[\frac{1}{2}] \rightarrow \mathbb{R}_* \mathbb{R}^*(\mathbf{cKW} \wedge \mathbf{cKW}[\frac{1}{2}])$. Let

$$\gamma^b: \Sigma^{s_j, 0} B(j) \wedge \mathbf{cKW}[\frac{1}{2}] \rightarrow \mathbb{R}_* \mathbb{R}^*(\mathbf{cKW} \wedge \mathbf{cKW}[\frac{1}{2}]) \simeq \mathbf{cKW} \wedge \mathbb{R}_* \mathbb{R}^*(\mathbf{cKW}[\frac{1}{2}]) \simeq \mathbf{cKW} \wedge \mathbf{cKW}[\frac{1}{2}]$$

be the composition of this adjoint map, the equivalence occurring in the projection formula (4) for the real Betti realization displayed in Theorem 3.1, and finally the equivalence $\mathbf{cKW}[\frac{1}{2}] \rightarrow \mathbb{R}_* \mathbb{R}^*(\mathbf{cKW}[\frac{1}{2}])$ from Corollary 4.9.

Corollary 7.2 *Let p be an odd prime. The map*

$$\gamma^b: \bigvee_{j \geq 0} \Sigma^{s_j, 0} B(j) \wedge \mathbf{cKW}[\frac{1}{2}] \rightarrow \mathbf{cKW} \wedge \mathbf{cKW}[\frac{1}{2}]$$

induced by the p -local equivalence (8) is a p -local equivalence in $\mathbf{SH}(\mathbb{R})$.

Proof The adjunction $(\mathbb{R}^*, \mathbb{R}_*)$ of integral stable homotopy categories descends to the “same” adjunction of p -local stable homotopy categories. Theorem 7.1 implies that the map (9) is a p -local equivalence. Its adjoint $\Sigma^{s_j, 0} B(j) \wedge \mathbf{cKW}[\frac{1}{2}] \rightarrow \mathbb{R}_* \mathbb{R}^*(\mathbf{cKW} \wedge \mathbf{cKW}[\frac{1}{2}])$ is computed as the image of (9) under \mathbb{R}_* , composed with the unit of the adjunction. Since \mathbb{R}_* preserves p -local equivalences, it suffices to show that the unit

$$\Sigma^{s_j, 0} B(j) \wedge \mathbf{cKW}[\frac{1}{2}] \rightarrow \mathbb{R}_* \mathbb{R}^* \Sigma^{s_j, 0} B(j) \wedge \mathbf{cKW}[\frac{1}{2}]$$

is a p -local equivalence. However, it is in fact an equivalence, by Theorem 4.9 and the projection formula from Theorem 3.1.

Corollary 7.2 implies the same p -local equivalence over any real closed field.

Theorem 7.3 *Let F be a field of characteristic zero. Then the map*

$$cu \wedge \mathbf{cKW}[\frac{1}{2}]: \mathbf{cKW}[\frac{1}{2}] \rightarrow \mathbf{cKW} \wedge \mathbf{cKW}[\frac{1}{2}]$$

is 3-connective. In particular, $\pi_{s,t} \mathbf{cKW} \wedge \mathbf{cKW}[\frac{1}{2}]$ is trivial for $1 \leq s - t \leq 3$.

Proof The source $\mathbf{cKW}[\frac{1}{2}]$ of the map $cu \wedge \mathbf{cKW}[\frac{1}{2}]$ satisfies real étale descent by [19, Theorem 6.6], and its target does so by Proposition 5.7. The statement follows by proving that the induced map

$$\pi_{s,t} \mathbf{cKW}[\frac{1}{2}] \rightarrow \pi_{s,t} \mathbf{cKW} \wedge \mathbf{cKW}[\frac{1}{2}] \tag{10}$$

on homotopy sheaves for the real étale topology is an isomorphism for all $s - t \leq 3$. Hence it suffices to prove that (10) is an isomorphism for all $s - t \leq 3$ after evaluating at a henselian local ring with real closed residue field. The rigidity property of Witt groups [20, Satz 3.3] provides that it suffices to evaluate (10) at real closed fields.

The main results of [2] imply that $cu \wedge \mathbf{cKW}_{\mathbb{Q}}: \mathbf{cKW}_{\mathbb{Q}} \rightarrow \mathbf{cKW} \wedge \mathbf{cKW}_{\mathbb{Q}}$ is 3-connective. More precisely, [2, Corollary 5] and [2, Theorem 5] imply that $\mathbf{cKW}_{\mathbb{Q}} \simeq \bigvee_{n \geq 0} \Sigma^{8n, 4n} \mathbf{1}[\frac{1}{\eta}]_{\mathbb{Q}}$ whence

$$\mathbf{cKW}_{\mathbb{Q}} \wedge \mathbf{cKW}_{\mathbb{Q}} \simeq \bigvee_{n \geq 0} \Sigma^{8n, 4n} \bigvee_{j=0}^n \mathbf{1}[\frac{1}{\eta}]_{\mathbb{Q}}.$$

In particular, $\pi_{s,t} \mathbf{cKW} \wedge \mathbf{cKW}_{\mathbb{Q}}$ is trivial for $1 \leq s - t \leq 3$. Hence it suffices to show that p -torsion vanishes in that range one odd prime p at a time. But this is an immediate consequence of Corollary 7.2 which holds for real closed fields.

More precise information on Witt cooperations, at least after inverting 2, can be deduced from knowledge about the topological spectra $B(j)$. They are given as Thom spectra of maps from even parts of the May filtration on the homotopy fiber of $\Omega^2 S^3 \rightarrow S^1$ to \mathbf{KO} ; see [22, Section 2] for details.

8 The vanishing

As mentioned already in Section 1, considering connective Witt theory with 2 inverted suffices due to knowledge on $(2, \eta)$ -completed groups from [31], which coincide with 2-completed groups under the following circumstances.

Theorem 8.1 (Hu-Kriz-Ormsby) *Let F be a field of finite virtual 2-cohomological dimension and of characteristic not two. Then the canonical map $\mathbf{1}_2^\wedge \rightarrow \mathbf{1}_{2,\eta}^\wedge$ is an equivalence.*

Proof The reference [17, Theorem 1] provides this statement in the case of fields of characteristic zero. Their proof goes through for fields of odd characteristic with the amendment that the motivic Eilenberg-MacLane spectrum $\mathbf{MZ}/2$ admits a cell presentation of finite type also over fields of characteristic not two [31, Proposition 3.32].

Lemma 8.2 *Let F a field of characteristic not two. The canonical map*

$$\pi_{n+1,n}\mathbf{1} \rightarrow \pi_{n+1,n}\mathbf{1}[\tfrac{1}{2}]$$

is injective for every natural number $n \geq 5$.

Proof Since the base field is a filtered colimit of fields of finite virtual 2-cohomological dimension, and $\pi_{n+1,n}\mathbf{1}$ commutes with filtered colimits of fields, one may assume that the base field has finite virtual 2-cohomological dimension. Consider the arithmetic square for 2 and $\mathbf{1}$. It induces a long exact sequence

$$\cdots \rightarrow \pi_{n+2,n}\mathbf{1}_2^\wedge[\tfrac{1}{2}] \rightarrow \pi_{n+1,n}\mathbf{1} \rightarrow \pi_{n+1,n}\mathbf{1}[\tfrac{1}{2}] \oplus \pi_{n+1,n}\mathbf{1}_2^\wedge \rightarrow \pi_{n+1,n}\mathbf{1}_2^\wedge[\tfrac{1}{2}] \rightarrow \cdots$$

of homotopy groups. The argument below will provide that $\pi_{n+2,n}\mathbf{1}_2^\wedge = 0$. Theorem 8.1 implies that the natural map $\pi_{s,t}\mathbf{1}_2^\wedge \rightarrow \pi_{s,t}\mathbf{1}_{2,\eta}^\wedge$ is an isomorphism for all s, t . The convergence result [31, Theorem 3.50] supplies a weak equivalence $\mathbf{sc}(\mathbf{1}) \simeq \mathbf{1}_\eta^\wedge$ between the η -completion and the completion with respect to the slice filtration of $\mathbf{1}$, hence an isomorphism $\pi_{s,t}\mathbf{1}_{2,\eta}^\wedge \cong \pi_{s,t}(\mathbf{sc}\mathbf{1})_2^\wedge$ for all s, t . Consider Milnor's derived limit exact sequence:

$$0 \rightarrow \lim_k^1 \pi_{s+1,t}(\mathbf{sc}\mathbf{1})/2^k \rightarrow \pi_{s,t}(\mathbf{sc}\mathbf{1})_2^\wedge \rightarrow \lim_k \pi_{s,t}(\mathbf{sc}\mathbf{1})/2^k \rightarrow 0 \quad (11)$$

The canonical map $(\mathbf{sc}\mathbf{1})/2^k \rightarrow \mathbf{sc}(\mathbf{1}/2^k)$ is an equivalence. As stated in the beginning of [31, Section 5], the computation of the first slice differential for the motivic sphere spectrum [31, Lemma 4.1] implies that $\pi_{n+1,n}\mathbf{sc}(\mathbf{1}) = 0$ for $n \geq 3$ and $\pi_{n+2,n}\mathbf{sc}(\mathbf{1}) = 0$ for $n \geq 5$. The long exact sequence of homotopy groups hence forces

$$\pi_{n+2,n}\mathbf{sc}(\mathbf{1})/2^k = 0 \quad \text{for } n \geq 5 \text{ and } k \geq 1$$

which – together with Milnor's short exact sequence (11) – implies the isomorphisms

$$\lim_k^1 \pi_{n+3,n}(\mathbf{sc}\mathbf{1})/2^k \xrightarrow{\cong} \pi_{n+2,n}(\mathbf{sc}\mathbf{1})_2^\wedge \quad \text{and} \quad \pi_{n+1,n}(\mathbf{sc}\mathbf{1})_2^\wedge \xrightarrow{\cong} \lim_k \pi_{n+1,n}(\mathbf{sc}\mathbf{1})/2^k.$$

It also implies that the canonical map $\pi_{n+3,n}(\mathbf{sc}\mathbf{1}) \rightarrow \pi_{n+3,n}(\mathbf{sc}\mathbf{1})/2^k$ is surjective for all $n \geq 5$ and all $k \geq 1$. Hence the canonical map $\pi_{n+3,n}(\mathbf{sc}\mathbf{1})/2^{k+1} \rightarrow \pi_{n+3,n}(\mathbf{sc}\mathbf{1})/2^k$ is surjective for all $k \geq 1$ by the commutative diagram

$$\begin{array}{ccc} \pi_{n+3,n}(\mathbf{sc}\mathbf{1}) & \longrightarrow & \pi_{n+3,n}(\mathbf{sc}\mathbf{1})/2^{k+1} \\ \text{id} \downarrow & & \downarrow \\ \pi_{n+3,n}(\mathbf{sc}\mathbf{1}) & \longrightarrow & \pi_{n+3,n}(\mathbf{sc}\mathbf{1})/2^k. \end{array}$$

In particular,

$$0 = \lim_k^1 \pi_{n+3,n}(\mathbf{sc}\mathbf{1})/2^k = \pi_{n+2,n}(\mathbf{sc}\mathbf{1})_2^\wedge = \pi_{n+2,n}\mathbf{1}_2^\wedge$$

for all $n \geq 5$. Thus the map

$$\pi_{n+1,n}\mathbf{1} \rightarrow \pi_{n+1,n}\mathbf{1}[\frac{1}{2}] \oplus \pi_{n+1,n}\mathbf{1}_2^\wedge$$

is injective for all $n \geq 5$. It remains to observe that the map $\pi_{n+1,n}\mathbf{1} \rightarrow \pi_{n+1,n}\mathbf{1}_2^\wedge$ is the zero map for all $n \geq 5$. In fact, the isomorphisms $\pi_{n+1,n}\mathbf{1}_2^\wedge \cong \pi_{n+1,n}\mathbf{sc}(\mathbf{1})_2^\wedge \cong \lim_k \pi_{n+1,n}(\mathbf{sc}\mathbf{1})/2^k$ explained above show that it suffices to prove the triviality of the canonical map $\pi_{n+1,n}\mathbf{1} \rightarrow \pi_{n+1,n}\mathbf{sc}(\mathbf{1})/2^k$ for all $n \geq 5$ and all $k \geq 1$, which again follows from the vanishing of $\pi_{n+2,n}\mathbf{sc}(\mathbf{1})$ for all $n \geq 5$.

The statement in Lemma 8.2 is not optimal. The computation [31, Theorem 5.5] which is based on Theorem 8.3 below implies that $\pi_{n+1,n}\mathbf{1} = 0$ for $n \geq 3$. The map $\pi_{3,2}\mathbf{1} \rightarrow \pi_{3,2}\mathbf{1}[\frac{1}{2}]$ is not injective; up to isomorphism it is the projection $\mathbb{Z}/24 \rightarrow \mathbb{Z}/3$ by [31, Theorem 5.5].

Theorem 8.3 *Let F be a field of characteristic not two. Then $\pi_{n+1,n}\mathbf{1}[\frac{1}{\eta}] = \pi_{n+2,n}\mathbf{1}[\frac{1}{\eta}] = 0$ for every integer n .*

Proof Lemma 8.2 (which holds on the level of homotopy sheaves, for example by [16, Theorem 2.7]) shows that it suffices to prove the statement for $\mathbf{1}[\frac{1}{\eta}, \frac{1}{2}]$. Thanks to Theorem 6.1, it is enough to consider fields of characteristic zero. Proposition 5.5 implies the isomorphism

$$\pi_{n+1,n}\mathbf{1}[\frac{1}{\eta}, \frac{1}{2}] \cong \pi_{n+2,n}\mathbf{cKW} \wedge \mathbf{cKW}[\frac{1}{2}]$$

for every integer n . Hence this sheaf of homotopy groups vanishes by Theorem 7.3. This in turn shows that $\mathbf{1}[\frac{1}{\eta}] \rightarrow \mathbf{cKW}$ is 2-connective, whence Proposition 5.6 implies an isomorphism

$$\pi_{n+2,n}\mathbf{1}[\frac{1}{\eta}, \frac{1}{2}] \cong \pi_{n+3,n}\mathbf{cKW} \wedge \mathbf{cKW}[\frac{1}{2}]$$

for every integer n . The latter sheaf again vanishes by Theorem 7.3.

Theorem 8.3 shows that Theorem 7.3 already holds before inverting two. The vanishing of homotopy sheaves in Theorem 8.3 implies the vanishing of the homotopy groups

$$\pi_{n+1,n}\mathbf{1}[\frac{1}{\eta}] = \pi_{n+2,n}\mathbf{1}[\frac{1}{\eta}] = 0$$

for all integers n stated in Theorem 1.1.

References

- [1] A. Ananyevskiy. Stable operations and cooperations in derived Witt theory with rational coefficients. [arXiv:1504.04848](#).
- [2] A. Ananyevskiy, M. Levine, and I. Panin. Witt sheaves and the η -inverted sphere spectrum. [arXiv:1504.04860](#).

- [3] M. Andrews and H. Miller. Inverting the Hopf map in the Adams-Novikov spectral sequence. <http://math.mit.edu/~mjandr/>, October 2014.
- [4] J. Ayoub. L’algèbre de Hopf et le groupe de Galois motiviques d’un corps de caractéristique nulle. I. *J. Reine Angew. Math.*, 693:1–149, 2014.
- [5] T. Bachmann. On the conservativity of some functors in motivic homotopy theory. [arXiv:1506.07375v2](#).
- [6] T. Bachmann. Motivic and real étale stable homotopy theory. [arXiv:1608.08855](#).
- [7] D. M. Davis. Odd primary bo -resolutions and K -theory localization. *Illinois J. Math.*, 30(1):79–100, 1986.
- [8] D. Dugger and D. C. Isaksen. Motivic cell structures. *Algebr. Geom. Topol.*, 5:615–652, 2005.
- [9] H. Fausk, P. Hu, and J. P. May. Isomorphisms between left and right adjoints. *Theory Appl. Categ.*, 11(4):107–131, 2003.
- [10] B. J. Guillou and D. C. Isaksen. The η -inverted \mathbb{R} -motivic sphere. *Algebr. Geom. Topol.*, 16:3005–3027, 2016.
- [11] B. J. Guillou and D. C. Isaksen. The η -local motivic sphere. *J. Pure Appl. Algebra*, 219(10):4728–4756, 2015.
- [12] J. J. Gutiérrez, O. Röndigs, M. Spitzweck, and P. A. Østvær. Motivic slices and coloured operads. *J. Topol.*, 5(3):727–755, 2012.
- [13] J. Heller and K. Ormsby. Galois equivariance and stable motivic homotopy theory. *Trans. Am. Math. Soc.*, 368(11):8047–8077, 2016.
- [14] J. Hornbostel. Some comments on motivic nilpotence. [arXiv:1511.07292](#).
- [15] J. Hornbostel. \mathbb{A}^1 -representability of hermitian K -theory and Witt groups. *Topology*, 44(3):661–687, 2005.
- [16] M. Hovey. From algebraic cobordism to motivic cohomology. *J. Reine Angew. Math.*, 702:173–226, 2015.
- [17] P. Hu, I. Kriz, and K. Ormsby. Convergence of the motivic Adams spectral sequence. *J. K-Theory*, 7(3):573–596, 2011.
- [18] R. M. Kane. Operations in connective K -theory. *Mem. Amer. Math. Soc.*, 34(254):vi+102, 1981.
- [19] M. Karoubi, M. Schlichting, and C. Weibel. The Witt groups of real algebraic varieties. [arXiv:1506.03862](#).

- [20] M. Knebusch. Isometrien über semilokalen Ringen. *Math. Z.*, 108:255–268, 1969.
- [21] W. Lellmann. Operations and co-operations in odd-primary connective K -theory. *J. London Math. Soc. (2)*, 29(3):562–576, 1984.
- [22] M. Mahowald. bo -resolutions. *Pacific J. Math.*, 92(2):365–383, 1981.
- [23] R. J. Milgram. The Steenrod algebra and its dual for connective K -theory. In *Conference on homotopy theory (Evanston, Ill., 1974)*, volume 1 of *Notas Mat. Simpos.*, pages 127–158. Soc. Mat. Mexicana, México, 1975.
- [24] Fabien Morel. On the motivic π_0 of the sphere spectrum. In *Axiomatic, enriched and motivic homotopy theory. Proceedings of the NATO Advanced Study Institute, Cambridge, UK, September 9–20, 2002*, pages 219–260. Kluwer Academic Publishers, 2004.
- [25] F. Morel. \mathbb{A}^1 -algebraic topology. In *International Congress of Mathematicians. Vol. II*, pages 1035–1059. Eur. Math. Soc., Zürich, 2006.
- [26] F. Morel. \mathbb{A}^1 -algebraic topology over a field, volume 2052 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012.
- [27] F. Morel and V. Voevodsky. \mathbb{A}^1 -homotopy theory of schemes. *Publ. Math., Inst. Hautes Étud. Sci.*, 90:45–143, 1999.
- [28] I. Panin and C. Walter. On the motivic commutative ring spectrum BO . [arXiv:1101.0650](https://arxiv.org/abs/1101.0650).
- [29] O. Röndigs and P. A. Østvær. Slices of hermitian K -theory and Milnor’s conjecture on quadratic forms. *Geometry & Topology* 20:1157–1212, 2016.
- [30] O. Röndigs, M. Spitzweck, and P. A. Østvær. Cellularity of hermitian K -theory and Witt theory. [arXiv:1603.05139](https://arxiv.org/abs/1603.05139). To appear in the TIFR Proceedings of the International Colloquium on K -theory 2016.
- [31] O. Röndigs, M. Spitzweck, and P. A. Østvær. The first motivic stable homotopy groups of spheres. [arXiv:1604.00365](https://arxiv.org/abs/1604.00365).
- [32] Winfried Scharlau. *Quadratic and Hermitian forms*, volume 270 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1985.
- [33] M. Schlichting. Hermitian K -theory, derived equivalences, and Karoubi’s fundamental theorem. To appear in *J. Pure Appl. Algebra* (2016), [http://dx.doi.org/10.1016/j.jpaa.2016.12.026](https://doi.org/10.1016/j.jpaa.2016.12.026).
- [34] M. Spivakovsky. A new proof of D. Popescu’s theorem on smoothing of ring homomorphisms. *J. Amer. Math. Soc.*, 12(2):381–444, 1999.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF OSNABRÜCK, OSNABRÜCK, GERMANY
E-mail: oliver.roendigs@uni-osnabrueck.de